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Force free projective motions of the sphere☆

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Abstract

Suppose that the sphere S^n has initially a homogeneous distribution of mass and let G be the Lie group of orientation preserving projective diffeomorphisms of S^n. A projective motion of the sphere, that is, a smooth curve in G, is called force free if it is a critical point of the kinetic energy functional. We find explicit examples of force free projective motions of S^n and, more generally, examples of subgroups H of G such that a force free motion initially tangent to H remains in H for all time (in contrast with the previously studied case for conformal motions, this property does not hold for $H = SO_{n+1}$). The main tool is a Riemannian metric on G, which turns out to be not complete (in particular not invariant, as happens with non-rigid motions), given by the kinetic energy.

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1. Introduction

The force free conformal motions of the sphere S^n have been studied by the second author in [7], with the aid of a suitable Riemannian metric on $SO_o(n+1, 1)$, an analogue of the classical description of the force free motions of a rigid body in Euclidean space using an invariant metric on $SO(3)$ [1, Appendix 2]. Other applications of the same technique to the study of the dynamics of a rigid body in the hyperbolic spaces of dimensions 2 and 3 can be found in [2,5,6]. In this note we define an appropriate metric on the special linear group $Sl(n+1, \mathbb{R})$ for studying force free projective motions of the sphere S^n. In some aspects the situation is similar to the conformal one, but there are also strong contrasts.

A diffeomorphism F of a Riemannian manifold M of dimension $n \geq 2$ is said to be *projective* if for any geodesic γ of M, $F \circ \gamma$ is a reparametrization (not necessarily of constant speed) of a geodesic of M. If M is oriented, a projective transformation of M will be called *directly projective* if it preserves the orientation.

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Let S^n be the unit sphere centered at zero in $\mathbb{R}^{n+1}$ and let G be the Lie group of directly projective diffeomorphisms of S^n. For $n \geq 2$, the directly projective transformations of S^n are exactly those of the form $p \mapsto Ap/|Ap|$ for some $A \in Gl_+(n+1, \mathbb{R})$ (we denote by $|Y|$ the canonical Euclidean norm of the vector Y). This was proved by Beltrami for $n = 2$ and for higher dimensions the same proof works: use the central projection of a hemisphere H to a hyperplane tangent at the center of H, and then the well-known fact that the projective transformations of $\mathbb{R}^n$ are the affine ones (see [4]). We thank Vladimir Matveev for having told us of the result and for the reference as well. By definition, the directly projective transformations of the circle S^1 are given by the canonical action of $Gl_+(2, \mathbb{R})$ on the circle, as above. Throughout the paper, smooth means of class C^∞.

1.1. The energy of projective motions

In this subsection and the next one we give some definitions and statements that are analogous to those given for the conformal (instead of the projective) situation in [7]. We present them here for the sake of completeness.

Suppose that the sphere has initially a homogeneous distribution of mass of constant density 1 and that the particles are allowed to move only in such a way that two configurations differ in an element of G. The configuration space may be naturally identified with G.

Let γ be a smooth curve in G, which may be thought of as a projective motion of S^n. The total kinetic energy $E(t)$ of the motion γ at the instant t is given by

$$E(t) = \frac{1}{2} \int_{S^n} \rho_t(q) \, |v_t(q)|^2 \, \mathrm{d}\mu(q), \tag{1}$$

where integration is with respect to the canonical volume form of S^n and, if $q = \gamma(t)(p)$ for $p \in S^n$, then

$$v_t(q) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_t \gamma(s)(p) \in T_q S^n, \qquad \rho_t(q) = 1/\det\left(d\gamma(t)_p\right)$$

are the velocity of the particle q and the density at q at the instant t, respectively. Applying to (1) the formula for change of variables, one obtains

$$E(t) = \frac{1}{2} \int_{S^n} \left| \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_t \gamma(s)(p) \right|^2 \mathrm{d}\mu(p). \tag{2}$$

The following definition is based on the principle of least action.

Definition. A smooth curve γ in G, thought of as a projective motion of S^n, is said to be *force free* if it is a critical point of the kinetic energy functional.

1.2. A Riemannian metric on the configuration space

Given $g \in G$ and $X \in T_g G$, let us define the map $\widetilde{X} : S^n \to TS^n$ by

$$\widetilde{X}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \gamma(t)(q) \in T_{g(q)} S^n, \tag{3}$$

where γ is any smooth curve in G with $\gamma(0) = g$ and $\dot{\gamma}(0) = X$. The map $\widetilde{X}$ is well defined and smooth and it is a vector field on S^n if and only if $X \in T_e G$. Moreover,

$$X \mapsto \|X\|^2 = \int_{S^n} \left|\widetilde{X}(q)\right|^2 \mathrm{d}\mu(q) \tag{4}$$

is a quadratic form on $T_g G$ and gives a Riemannian metric on G. The verification of the analogous assertions in the conformal case can be found in [7].

Remarks. (a) The fundamental property of the metric (4) on G is that a curve γ in G is a geodesic if and only if (thought of as a projective motion) it is force free, since one can verify easily that $E(t) = \frac{1}{2} \|\dot{\gamma}(t)\|^2$.

(b) For any n, the metric on G is neither left nor right invariant, since we will see in Theorem 2 below that it is not even complete.

1.3. Force free projective motions

We fix $n \geq 1$ and identify $G = Sl(n+1, \mathbb{R})$, which acts transitively and effectively on S^n through $A \cdot p = Ap/|Ap|$. These are exactly the directly projective transformations of S^n, as we mentioned above.

A great sphere of S^n is the intersection of S^n with a subspace of $\mathbb{R}^{n+1}$. A flag of orthogonal great spheres in S^n is a set $\{S_i \mid i = 0, \ldots, l\}$, where $S_i = S^n \cap V_i$ and V_i $(i = 0, \ldots, l)$ are nontrivial orthogonal subspaces of $\mathbb{R}^{n+1}$ whose union generates $\mathbb{R}^{n+1}$. Let $\{e_0, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^{n+1}$ and let $\mathbb{C}$, $\mathbb{H}$ denote the normed division algebras of the complex numbers and quaternions, respectively.

Theorem 1. *The following subgroups of G are totally geodesic:*

(a) *The set of directly projective transformations of S^n fixing a flag of orthogonal great spheres, in particular, those fixing the points of the canonical basis of $\mathbb{R}^{n+1}$.*

(b) *For $n = 2m - 1$, $GL_1(m, \mathbb{C}) = \{A \in M(m, \mathbb{C}) \mid |\det_{\mathbb{C}} A| = 1\}$, with its canonical inclusion in G.*

(c) *For $n = 4m - 1$, $Sl(m, \mathbb{H}) = \{A \in M(m, \mathbb{H}) \mid \det_{\mathbb{R}} A = 1\}$, with its canonical inclusion in G.*

Theorem 2. *Let $\alpha : (0, \infty) \to G$ be the curve in G defined by $\alpha(t) = \mathrm{diag}(1/t^n, t, \ldots, t)$. Then any reparametrization of α by arc length is a geodesic, has finite length and is inextendible. In particular, G is not a complete Riemannian manifold.*

Let $K = SO(n+1)$ be the group of orientation preserving isometries of S^n and let $\mathfrak{k}$ denote its Lie algebra.

Theorem 3. (a) *$K \times K$ acts on G on the left, $(h, k) \cdot g = hgk^{-1}$, by isometries of G. In particular, the metric on K induced from G is bi-invariant and hence its geodesics through the identity are one-parameter subgroups.*

(b) *For $Z \in \mathfrak{k}$, the geodesic $\sigma(t) = \exp(tZ)$ of K through the identity is also a geodesic of G if and only if $Z = \lambda J$ for some $\lambda \in \mathbb{R}$ and $J \in \mathfrak{k}$ with $J^2 = -I$. In particular, if n is even, no geodesic of K through the identity (except the constant one) is a geodesic of G.*

Corollary 4. *The flow of a Killing field V on S^n is a geodesic of G if and only if n is odd and V is a Hopf vector field.*

Remark. Part (b) of Theorem 3 contrasts strongly with the conformal situation in [7], where it is proved that K is totally geodesic in the group of directly conformal transformations of S^n endowed with the kinetic energy metric.

2. Proofs of the theorems

Proposition 5. *Let f, g be isometries of S^n, not necessarily in G, whose determinants are equal; in particular, $fhg \in G$ for all $h \in G$. Then $f(h \cdot g(q)) = (fhg) \cdot q$ for all $q \in S^n$ and $F_{f,g} : G \to G$, defined by $F_{f,g}(h) = fhg$, is an isometry of G with respect to the metric given in (4).*

The proof is similar to that of the analogous statement in the conformal situation in [7] and we omit it. In the case when $g = f^{-1}$ we call $F_{f,g}$ just F_f.

Corollary 6. *If f is an isometry of S^n, then the set of fixed points of F_f is a totally geodesic submanifold of G.*

Proof. It is well known that each connected component of the set of fixed points of an isometry is a totally geodesic submanifold. Now, the set of the fixed points of F_f is a Lie subgroup of G, in particular a submanifold. Hence, it is totally geodesic submanifold of G. □

Proof of Theorem 1. (a) We may suppose that the flag of great spheres is given by the intersection of S^n with subspaces $V_i = \mathrm{span}\ \{e_k \mid k_i \leq k < k_{i+1}\}$, $0 \leq i \leq l$, where $0 = k_0 < k_1 < \cdots < k_{l+1} = n + 1$ and $0 \leq l \leq n$. For $1 \leq i \leq l$ let $f_i = \mathrm{diag}\,(-I_{k_i}, I_{n+1-k_i})$ and let H_i be the set of fixed points of F_{f_i}, which consists of the matrices of G of the form $\mathrm{diag}\,(A, B)$, with A, B square matrices with k_i and $n + 1 - k_i$ rows, respectively. By Corollary 6, H_i, which is the subgroup of G fixing the subspaces span $\{e_0, \ldots, e_{k_i-1}\}$ and span $\{e_{k_i}, \ldots, e_n\}$, is totally geodesic in G. So $\cap_{i=1}^{l} H_i$, the subgroup fixing the given flag of great spheres, is also totally geodesic in G. This holds in particular when $l = n$ and all the great spheres of the flag have dimension zero. Thus, (a) is proved. For $m \in \mathbb{N}$ we define

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & -I_{2m} \\ I_{2m} & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}. \tag{5}$$

(b) There is a canonical monomorphism $\iota : Gl(m, \mathbb{C}) \to Gl(n+1, \mathbb{R})$, whose image consists of the elements of the latter commuting with J. Now, we have by [3, p 16] that $|\det_{\mathbb{C}}(A)|^2 = \det_{\mathbb{R}} \iota(A)$ for any $A \in Gl(m, \mathbb{C})$. Thus, $Gl_1(m, \mathbb{C})$ is the set of fixed points of F_J defined on G, and hence totally geodesic in G by Corollary 6.

(c) We identify as usual $Gl(m, \mathbb{H})$ with the set of elements in $Gl_+(n+1, \mathbb{R})$ commuting with both J_1 and J_2 as in (5). So $Sl(m, \mathbb{H})$ is the intersection of the sets of fixed points of F_{J_1} and F_{J_2} defined on G, and hence totally geodesic, again by Corollary 6. $\square$

Proof of Theorem 2. By Theorem 1(a), the set of diagonal matrices in G is totally geodesic, since it is the subgroup of G fixing the complete flag of zero-dimensional great spheres $S_i = \{e_i, -e_i\}$, $0 \le i \le n$. So, its connected component of the identity, $D = \{\mathrm{diag}(a_0, \ldots, a_n) \mid a_i > 0\}$, is also totally geodesic in G. If $n = 1$, $D = \alpha(0, \infty)$. Now, if $n > 1$, let $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $R_i = \mathrm{diag}(I_i, R, I_{n-1-i})$, for $1 \le i < n$. By Corollary 6, the set D_i of fixed points of F_{R_i}, which consists of the elements of D as above with $a_i = a_{i+1}$, is totally geodesic in G. Now, an easy computation shows that $\alpha(0, \infty) = \cap_{i=1}^{n-1} D_i$, so it is totally geodesic in G. Hence the first assertion is immediate. In order to check the validity of the second statement, we recall from (3) and (4) that

$$\|\dot{\alpha}(t)\|^2 = \int_{S^n} \left|\widetilde{X}_t(q)\right|^2 \mathrm{d}\mu(q),$$

where $\widetilde{X}_t(q) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \gamma(s) \cdot q$, with $\gamma(0) = \alpha(t)$ and $\dot{\gamma}(0) = \dot{\alpha}(t)$. We can take

$$\gamma(s) = \alpha(s+t) = \mathrm{diag}\left(1/(s+t)^n, (s+t) I_n\right).$$

Let $q \in S^n$ with $q = (x_0, x)$, $x = (x_1, \ldots, x_n)$. Then

$$\gamma(s) \cdot q = \left(x_0, (s+t)^{n+1} x\right) \Big/ \sqrt{x_0^2 + (s+t)^{2n+2} |x|^2}.$$

Straightforward computations give

$$\widetilde{X}_t(q) = t^n (n+1) x_0 \left(-t^{n+1} |x|^2, x_0 x\right) \Big/ \left(x_0^2 + t^{2n+2} |x|^2\right)^{3/2},$$

$$\left|\widetilde{X}_t(q)\right|^2 = (n+1)^2 x_0^2 t^{2n} |x|^2 \Big/ \left(x_0^2 + t^{2n+2} |x|^2\right)^2.$$

To integrate we change variables: $F : (-\pi/2, \pi/2) \times S^{n-1} \to S^n$, $F(\theta, y) = (\sin\theta, y\cos\theta) = (x_0, x)$. The Jacobian factor is $\cos^{n-1}\theta$; hence

$$\begin{aligned}
\|\dot{\alpha}(t)\|^2 &= \int_{S^n} \left|\widetilde{X}_t(q)\right|^2 \mathrm{d}\mu(q) \\
&= \int_{-\pi/2}^{\pi/2} \int_{S^{n-1}} \frac{(n+1)^2 t^{2n} \sin^2\theta \, |y\cos\theta|^2}{\left(\sin^2\theta + t^{2n+2} |y\cos\theta|^2\right)^2} \cos^{n-1}\theta \mathrm{d}\nu(y) \, \mathrm{d}\theta \\
&= 2(n+1)^2 \mathrm{vol}\left(S^{n-1}\right) \int_0^{\pi/2} \frac{t^{2n} \sin^2\theta \cos^{n+1}\theta}{\left(\sin^2\theta + t^{2n+2}\cos^2\theta\right)^2} \mathrm{d}\theta \\
&\le (n+1)^2 \mathrm{vol}\left(S^{n-1}\right) \int_0^{\pi/2} \frac{t^{2n} 2\sin\theta\cos\theta}{\left(1 - \cos^2\theta + t^{2n+2}\cos^2\theta\right)^2} \mathrm{d}\theta.
\end{aligned}$$

Now we substitute $u = \cos^2\theta$ and obtain that $\|\dot\alpha(t)\|^2 \le c_n/t^2$ where c_n depends only on n. Therefore

$$\left(\text{length}\left(\alpha|_{[1,\infty)}\right)\right)^2 \le \int_1^\infty \|\dot\alpha(t)\|^2\, \mathrm{d}t < \infty.$$

Clearly $\lim_{t\to\infty}\alpha(t)$ does not exist; hence α cannot be extended. $\square$

Let $\mathfrak{k} = so\,(n+1)$ be as above the Lie algebra of K and let $\mathfrak{p}$ be the subspace of symmetric matrices in $\mathfrak{g}$, the Lie algebra of G.

The statement of the following proposition is analogous to the one in the conformal situation in [7], but the proof is different and we include it.

Proposition 7. *With respect to the metric on G defined in (4),* $\langle\mathfrak{k},\mathfrak{p}\rangle = 0$.

Proof. For any $Y \in \mathfrak{g} \cong T_eG$ and $q \in S^n$ we have

$$\widetilde{Y}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{tY}\cdot q = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{tY}q/\left|\mathrm{e}^{tY}q\right| = Yq - \langle Yq, q\rangle\, q.$$

Let $Z \in \mathfrak{k}$, $X \in \mathfrak{p}$. Since Z is skew symmetric, $\langle Zq, q\rangle = 0$, and we compute

$$\langle Z, X\rangle = \int_{S^n}\left\langle\widetilde{Z}(q), \widetilde{X}(q)\right\rangle = \int_{S^n}\langle Zq, Xq\rangle = \sum_{i,j,\ell} Z_{ij}X_{i\ell}\int_{S^n} x_jx_\ell,$$

if $q = (x_0,\ldots,x_n)$. Now, $\int_{S^n} x_jx_\ell = 0$ if $j \ne \ell$, since x_jx_ℓ is an odd function on S^n with respect to the reflection fixing $e_j^\perp$. Therefore

$$\langle Z, X\rangle = \sum_{i,j} Z_{ij}X_{ij}\int_{S^n} x_j^2 = \sum_{i,j} Z_{ij}X^t_{ji}\int_{S^n} x_0^2 = \operatorname{tr}\left(ZX^t\right)\int_{S^n} x_0^2,$$

which vanishes since $X^t = X$ and $Z + Z^t = 0$ imply that $\operatorname{tr}\left(ZX^t\right) = 0$. $\square$

Lemma 8. *Let $f(t,s)$ be a smooth never vanishing function from an open neighborhood of 0 in $\mathbb{R}^2$ to a vector space, such that $\langle f(0), f_s(0)\rangle = 0$ and $|f(0)| = 1$, and let $h = f/|f|$; then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 |h_s(t,0)|^2 = 2\langle f_{s,t}(0), f_s(0)\rangle - 2\langle f_t(0), f(0)\rangle\, |f_s(0)|^2, \tag{6}$$

where the subindexes s, t denote partial derivatives with respect to s, t.

Proof. One computes that $|h_s|^2 = A - B$, where $A = |f_s|^2/|f|^2$ and $B = \left(\langle f_s, f\rangle/|f|^2\right)^2$. Further straightforward computations, using the conditions on f, yield that $(\mathrm{d}/\mathrm{d}t)|_0\, B(t,0) = 0$ and $(\mathrm{d}/\mathrm{d}t)|_0\, A(t,0)$ equals the right hand side of (6). Thus, the lemma follows. $\square$

Proof of Theorem 3. The proof of (a) is immediate from Proposition 5. Now we prove (b). The proof will not be very simple because of the fact that the metric on G is not invariant. Let us define $\gamma_Z(t) = \exp(tZ)$. By (a), writing $\gamma_Z(t+t_o) = \gamma_Z(t_o)\,\gamma_Z(t)$, we have that γ_Z is a geodesic in G if and only if $(\nabla_Z Z)_e = 0$, or equivalently, $\left\langle(\nabla_Z Z)_e, Y\right\rangle = 0$ for all $Y \in \mathfrak{g}$. By the formula for the Levi-Civita connection we have

$$2\left\langle(\nabla_Z Z)_e, Y_e\right\rangle = 2Z_e\langle Y, Z\rangle - 2\langle[Z,Y]_e, Z_e\rangle - Y_e\|Z\|^2. \tag{7}$$

Now, since $K\times K$ acts by isometries on G by (a), we have

$$\langle Y, Z\rangle_{\exp(tZ)} = \left\langle \mathrm{d}L_{\exp(tZ)}Y_e, \mathrm{d}L_{\exp(tZ)}Z_e\right\rangle = \langle Y_e, Z_e\rangle$$

for all t. Hence the first term of the right hand side of (7) is zero. Now write $Y = Z' + X$, with $Z' \in \mathfrak{k}$ and $X \in \mathfrak{p}$. Again by the K-invariance, $\left\langle\left[Z, Z'\right]_e, Z_e\right\rangle$ and $Z'_e\|Z\|^2$ both vanish. Finally, $\langle[Z,X]_e, Z_e\rangle = 0$ by Proposition 7, since $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition and so $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$. Consequently, by (7), γ_Z is a geodesic in G if and only if $X_e\|Z\|^2 = 0$ for all $X \in \mathfrak{p}$.

Define $U_t = Z(\exp(tX)) = dL_{\exp(tX)}(Z_e)$. Fix momentarily $q \in S^n$ and let $f(t,s) = \exp(tX)\exp(sZ)\,q$ and $h = f/|f|$. Now, $f(0) = q$, $f_t(0) = Xq$, $f_s(0) = Zq$ and $f_{s,t}(0) = XZq$. Since Z is skew symmetric, f satisfies the hypotheses of Lemma 8 and so we have by (6) that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left|\frac{\partial h}{\partial s}(t,0)\right|^2 = 2\langle XZq, Zq\rangle - 2\langle Xq, q\rangle\,|Zq|^2 .$$

On the other hand,

$$X_e\left(\|Z\|^2\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \|Z(\exp(tX))\|^2 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{S^n} \left|\widetilde{U}_t(q)\right|^2 \mathrm{d}\mu(q) .$$

Now, since $\widetilde{U}_t(q) = \frac{\partial h}{\partial s}(t,0)$ (take $\gamma(s) = \mathrm{e}^{tX}\mathrm{e}^{sZ}$ in (3)), we have that

$$X_e\left(\|Z\|^2\right) = 2\int_{S^n} \langle XZq, Zq\rangle - \langle Xq, q\rangle\,|Zq|^2\,\mathrm{d}\mu(q) . \tag{8}$$

Clearly we may consider Z in $\mathfrak{k}$ up to multiples. Suppose first that $Z^2 = -\mathrm{id}$. We can distribute the integral and, since Z is orthogonal, change variables $p = Zq$ in the first term, with Jacobian factor 1. Hence $X_e\left(\|Z\|^2\right) = 0$, since $|Zq| = 1$.

Now suppose that $Z^2 \neq \lambda$ id for any λ. After conjugating by an element of K and multiplying by a suitable constant, we may suppose that either

$$Z = Z_0 = \mathrm{diag}(J_o, 0, B_0) \quad \text{or} \quad Z = Z_1 = \mathrm{diag}(J_o, aJ_o, B_1),$$

where $J_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B_0 \in so_{n-2}$, $B_1 \in so_{n-3}$ and $|a| \neq 1$ (the last case only if $n \geq 3$). Let $X = \mathrm{diag}(0, 1, -1, 0_{n-2}) \in \mathfrak{p}$. First we consider the case $Z = Z_0$ and show that $X_e\|Z_0\|^2 \neq 0$, and hence γ_{Z_0} is not a geodesic in G. If $q = (x_0, x_1, x_2, x)$, one computes

$$\langle XZ_0q, Z_0q\rangle = x_0^2, \qquad \langle Xq, q\rangle = x_1^2 - x_2^2, \qquad |Z_0q|^2 = x_0^2 + x_1^2 + |B_0x|^2 .$$

Therefore the integrand of (8) equals

$$x_0^2 - \left(x_1^2 - x_2^2\right)x_1^2 - \left(x_1^2 - x_2^2\right)\left(x_0^2 + |B_0x|^2\right),$$

whose last term is an odd function on S^n with respect to the reflection fixing the hyperplane $x_1 = x_2$. Hence its integral over the sphere is zero and consequently,

$$X_e\|Z_0\|^2 = 2\int_{S^n} x_0^2 - x_1^4 + x_2^2x_1^2 = 2\int_{S^n} x_0^2\left(1 - x_0^2 + x_1^2\right) > 0,$$

since clearly $\int_{S^n} x_1^4 = \int_{S^n} x_0^4$ and $\int_{S^n} x_2^2x_1^2 = \int_{S^n} x_0^2x_1^2$, and also $|x_0| \leq 1$.

Similar computations yield $X_e\|Z_1\|^2 = \left(1 - a^2\right)X_e\|Z_0\|^2$, which does not vanish since $|a| \neq 1$. $\square$

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